An Introduction to Solving Spherical Triangles

Any mathematician worth his salt is capable of solving triangles in the plane using a variety of methodologies: the definition of trigonometric ratios and their inverses, the laws of sines and cosines, and possibly even the Pythagorean theorem. Very few are versed in the art of solving triangles on the sphere, as the art of spherical geometry and spherical trigonometry is all but lost, due to the advent of many technologies which render the necessity to apply these fields obsolete.

Spherical Lines, Segments and Triangles

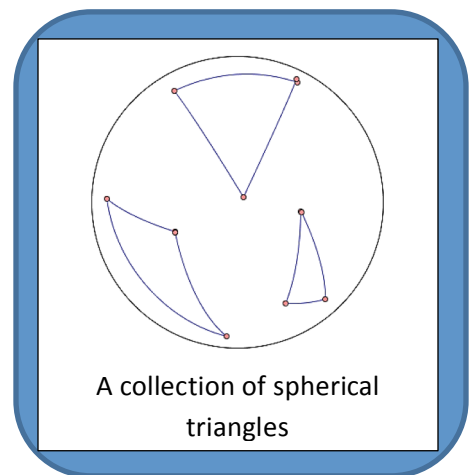
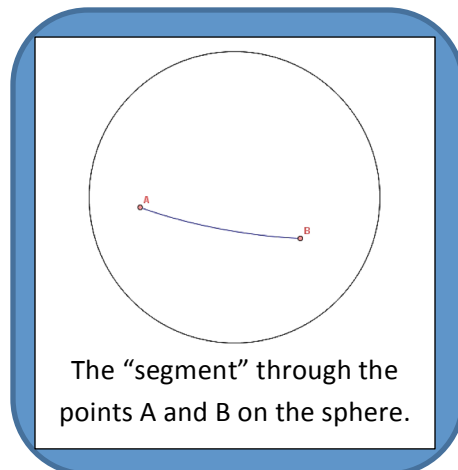
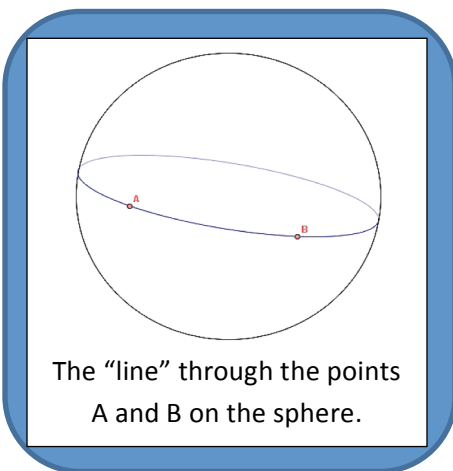
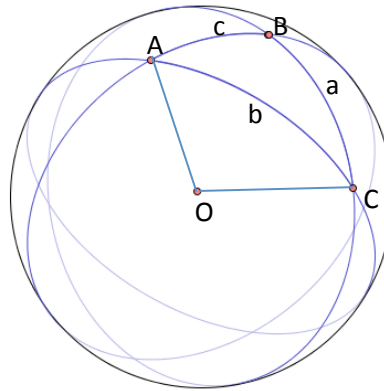
Consider, for a moment, any sphere. In spherical geometry and trigonometry, a line is defined as the intersection of a plane with the sphere, provided the plane passes through the sphere's center. In some circles (pun intended) it is known as a great circle.

A spherical segment, similarly, is a finite piece of such a spherical line. Any spherical segment must be incident with a great circle.

A spherical triangle, then, is a group of three spherical line segments, where the endpoints are incident with exactly two spherical segments. One can see that the triangles can take a variety of shapes, dependent upon how those segments interact with each other.

The "line" through the points A and B on the sphere.

The "segment" through the points A and B on the sphere.

A collection of spherical triangles

In spherical geometry, the spherical angle(henceforth denoted angle) formed by two spherical segments (hereafter referred to as segments) is measured by the angle formed by the planes intersecting the sphere to form the segments. Some prefer to think of the angle in terms of the tangent lines of the arcs of the great circles at the point in which they meet (their vertices). The segments themselves can also be thought of angularly, as they correspond to a central angle from the center of the sphere. If the sphere has a radius of 1 unit, then the segments will have a radian measure equivalent to that central angle.

Spherical Analogues to Planar Trigonometry

As is true in the plane, there is an correlating Law of Sines in the plane, which is that the sines of the sides of a spherical triangle are proportional to the sines of their opposite angles, or:

The Spherical Law of Sines:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

As well, there is a parallel to the planar Law of Cosines, which is: the cosine of any side of a spherical triangle is equal to the product of the cosines of the remaining two sides added to the product of the sines of those two sides multiplied by the cosine of their included angle, or:

The Spherical Law of Cosines:

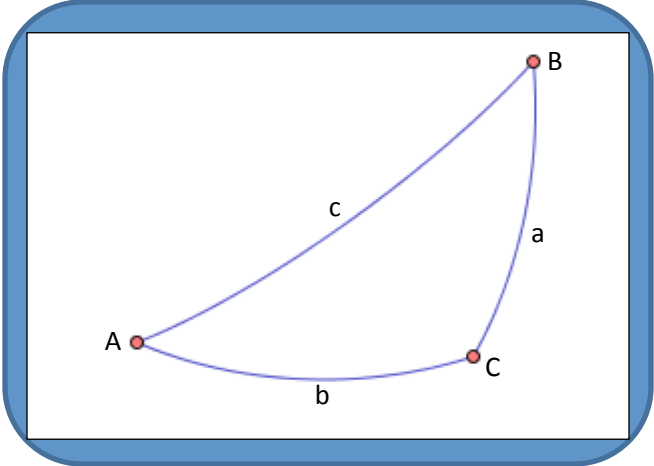
$$\cos a = \cos b \cos c + \sin a \sin b \cos A$$

A great many spherical triangles can be solved using these two laws, but unlike planar triangles, some require additional techniques known as the supplemental Law of Cosines, Napier's Rules (or Napier's Analogies), and the four part Cotangent Formulae. We shall focus our attention in this article on using the Spherical Law of Cosines (three sides known or two sides with the included angle known).

Solving Spherical Triangles with the Law of Cosines

Consider the case where we know the measurements for a, b, and C:

Let a = $76^{\circ}24'40''$, b = $58^{\circ}18'36''$ and C = $118^{\circ}30'28''$

Using the Spherical Law of Cosines, we can solve for c:

$\cos c = (\cos 76.41111^{\circ})(\cos 58.31^{\circ}) + (\sin 76.41111^{\circ})(\sin 58.31^{\circ})(\cos 118.50778^{\circ})$

$\cos c = -0.27132 \rightarrow c = \arccos(-0.27132) = 105.74295^{\circ} = 105^{\circ}44'35''$

Now, having an angle pair, we can continue by using the Spherical Law of Sines to solve for the remaining Spherical Angles

$$\frac{\sin 76.41111^{\circ}}{\sin A} = \frac{\sin 58.31^{\circ}}{\sin B} = \frac{\sin 118.50778^{\circ}}{\sin 105.74295^{\circ}}$$

It appears that for some triangles on the sphere, the methodologies for solving are quite similar if given the correct tools (formulas) in which to do so. While this essay has only scratched the surface of solving such triangular puzzles, it should give the reader some introductory background and knowledge if he cares to pursue this further. The reader may be interested in exploring the sphere further with a free downloadable program known as Spherical Easel (drawings in this writing were generated via this software) from http://merganser.math.gvsu.edu/easel/download.html.

Sources:

http://en.wikipedia.org/wiki/Spherical_trigonometry

Kells, Kern & Bland. *Plane and Spherical Trigonometry*. York, PA: Maple Press Company, 1935. Print.

Van Brummelen. *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry*. Princeton, NJ: Princeton University Press, 2013. Print.